Solutions 3

Exercise 3.19

According to (3.43) in example 3.15 of Page 59, we can get

$$\mathrm{var}(\hat{\beta}) \geq \frac{12}{(2\pi)^2 M\eta \frac{M+1}{M-1}(\frac{L}{\lambda})^2 \sin^2\beta}$$

where $\eta = 1$ $L = (M-1)\lambda/2$ and $\beta = 90°$. Thus,

$$\mathrm{var}(\hat{\beta}) \geq \frac{48}{(2\pi)^2 M(M^2-1)}$$

Since $\sqrt{\mathrm{var}(\hat{\beta})} = 5\pi/180$, we can obtain

$$M(M^2-1) \geq 159.7 \Longrightarrow M \geq 6$$

However, in this case, $L = (M-1)\lambda/2 = (M-1)c/(2F_0) = 750\text{m}$ where $F_0 = 10^6$. It is obviously impossible.

Exercise 3.20

Using (3.16) in Page 37 and 38, we can get

$$\mathrm{var}(\hat{P}_{xx}(f)) \geq \frac{(\frac{\partial P_{xx}(f)}{\partial a[1]})^2}{I(a[1])} = \frac{(\partial P_{xx}(f)/\partial a[1])^2}{N/(1-a^2[1])}$$

According to example 3.16, we have $P_{xx}(f) = \frac{\sigma_u^2}{|A(f)|^2}$ where $A(f) = 1 + a[1]e^{-j2\pi f}$. Thus,

$$\begin{aligned}\frac{\partial P_{xx}(f)}{\partial a[1]} &= \sigma_u^2 \cdot \frac{\partial}{\partial a[1]}\left(\frac{1}{A(f)A^*(f)}\right) \\ &= -\frac{2\sigma_u^2}{|A(f)|^4}(a[1] + \cos 2\pi f)\end{aligned}$$

Then,

$$\mathrm{var}(\hat{P}_{xx}(f)) \geq \frac{4\sigma_u^2(1-a^2[1])(a[1]+\cos 2\pi f)^2}{N|A(f)|^8}$$

For the given value, we can get

$$\mathrm{var}(\hat{P}_{xx}(f)) \geq \frac{0.0076(-0.9+\cos 2\pi f)^2}{|1-0.9e^{-j2\pi f}|^8}$$

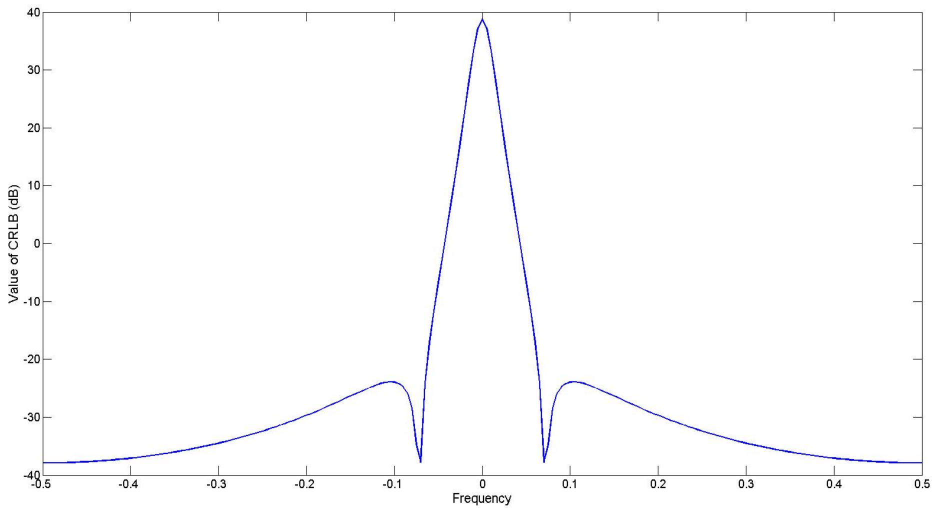

Figure 1: CRLB versus frequency

The variance is highest when $f = 0$ because of the sensitivity of the PSD to direct change in $a[1]$ for f near zero.

Exercise 5.3

$$p(\mathbf{x};\lambda) = \prod_{n=0}^{N-1} \lambda \exp\left(-\lambda x[n]\right) \qquad x[n] > 0$$
$$= \lambda^N \exp\left(-\lambda \sum_{n=0}^{N-1} x[n]\right) \cdot u(\min x[n])$$

Thus, $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]$ is a sufficient statistic.

Exercise 5.4

$$p(\mathbf{x};\lambda) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}(x[n]-\theta)^2}$$
$$= \frac{1}{(2\pi\theta)^{N/2}} e^{-\frac{1}{2\theta}\sum_{n=0}^{N-1}(x[n]-\theta)^2}$$
$$= \frac{1}{(2\pi\theta)^{N/2}} e^{-\frac{1}{2\theta}\sum_{n=0}^{N-1} x^2[n] - \frac{1}{2}N\theta} \cdot e^{\sum_{n=0}^{N-1} x[n]}$$

Thus, $T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n]$ is a sufficient statistic.

Exercise 5.5

$$p(x[n];\theta) = \frac{1}{2\theta}\left(u(x[n]+\theta) - u(x[n]-\theta)\right)$$
$$p(\mathbf{x};\theta) = \frac{1}{(2\theta)^N} \prod_{n=0}^{N-1} \left(u(x[n]+\theta) - u(x[n]-\theta)\right)$$

The product is zero unless $-\theta \leq x[n] \leq \theta$ for all $x[n]$ or $\max |x[n]| \leq \theta$, so that

$$p(\mathbf{x};\theta) = \frac{1}{(2\theta)^N} u(\theta - \max |x[n]|)$$

Thus, $T(\mathbf{x}) = \max |x[n]|$ is a sufficient statistic.

Exercise 5.14

(a)

$$p(x;\mu) = \exp\left[\mu x - \frac{1}{2}x^2 + \left(-\frac{1}{2}\mu^2 + \ln \frac{1}{\sqrt{2\pi}}\right)\right]$$

where $A(\mu) = \mu$, $B(x) = x$, $C(x) = -\frac{1}{2}x^2$ and $D(\mu) = -\frac{1}{2}\mu^2 + \ln \frac{1}{\sqrt{2\pi}}$.

(b)

$$\begin{aligned} p(x;\sigma^2) &= \frac{x}{\sigma^2} \exp\left[-\frac{1}{2}\frac{x^2}{\sigma^2}\right] u(x) \\ &= \exp\left[-\frac{1}{2}\frac{x^2}{\sigma^2} + \ln x u(x) - \ln \sigma^2\right] \end{aligned}$$

where $A(\sigma^2)B(x) = -\frac{1}{2}\frac{x^2}{\sigma^2}$, $C(x) = \ln x u(x)$ and $D(\sigma^2) = -\ln \sigma^2$.

(c)

$$p(x;\lambda) = \exp\left[-\lambda x + \ln u(x) + \ln \lambda\right]$$

where $A(\lambda)B(x) = -\lambda x$, $C(x) = \ln u(x)$ and $D(\lambda) = \ln \lambda$.

Exercise 5.15

$$\begin{aligned} p(\mathbf{x},\theta) &= \prod_{n=0}^{N-1} \exp[A(\theta)B(x[n]) + C(x[n]) + D(\theta)] \\ &= \exp[A(\theta)\sum_n B(x[n]) + \sum_n C(x[n]) + ND(\theta)] \\ &= \exp[A(\theta)\sum_n B(x[n]) + ND(\theta)] \exp[\sum_n C(x[n])] \end{aligned}$$

Thus, $T(x) = \sum_{n=0}^{N-1} B(x[n])$ is a sufficient statistic.

For (a), $T(x) = \sum_n x[n]$, for (b), $T(x) = \sum_n x^2[n]$ and for (c), $T(x) = \sum_n x[n]$.

For Gaussian, the MVU is $\hat{\mu} = \frac{1}{N}T(x) = \frac{1}{N}\sum_n x[n]$.

For Rayleigh, we have $\mathbb{E}(x^2) = 2\sigma^2$. Thus, $\hat{\sigma^2} = \frac{1}{2N}\sum_n x^2[n]$.

For exponential, we have $\mathbb{E}(x) = \frac{1}{\lambda}$.Denote $\nu = \frac{1}{\lambda}$, then $\hat{\nu} = \frac{1}{N}\sum_n x[n]$.